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Author(s): Peter A. Diamond
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NATIONAL DEBT IN A NEOCLASSICAL GROWTH MODEL

By PETER A. DIAMOND*

This paper contains a model designed to serve two purposes, to examine long-run competitive equilibrium in a growth model and then to explore the effects on this equilibrium of government debt. Samuelson [8] has examined the determination of interest rates in a single-commodity world without durable goods. In such an economy, interest rates are determined by consumption loans between individuals of different ages. By introducing production employing a durable capital good into this model, one can examine the case where individuals provide for their retirement years by lending to entrepreneurs. After describing alternative long-run equilibria available to a centrally planned economy, the competitive solution is described. In this economy, which has an infinitely long life, it is seen that, despite the absence of all the usual sources of inefficiency, the competitive solution can be inefficient.

Modigliani [4] has explored the effects of the existence of government debt in an aggregate growth model. By introducing a government which issues debt and levies taxes to finance interest payments into the model described in the first part, it is possible to re-examine his conclusions in a model where consumption decisions are made individually, where taxes to finance the debt are included in the analysis, and where the changes in output arising from changes in the capital stock are explicitly acknowledged. It is seen that in the "normal" case external debt reduces the utility of an individual living in long-run equilibrium. Surprisingly, internal debt is seen to cause an even larger decline in this utility level.

External debt has two effects in the long run, both arising from the taxes needed to finance the interest payments. The taxes directly reduce available lifetime consumption of the individual taxpayer. Further, by reducing his disposable income, taxes reduce his savings and thus the capital stock. Internal debt has both of these effects as well as a further reduction in the capital stock arising from the substitution of government debt for physical capital in individual portfolios.

* The author is assistant professor of economics at the University of California, Berkeley. He wishes to thank his colleagues Bernard Saffran and Sidney G. Winter, Jr. for many helpful discussions on this subject. Errors and interpretations are solely his own.

1. Technology

The economy being considered here is assumed to have an infinite future. Its unchanging technology is assumed to be representable by a constant returns to scale aggregate production function, $F(K, L)$.[1] Since the economy exists in discrete time, the capital argument of the production function is the saving of the previous period plus the capital stock employed in the previous period. (It is assumed that there is no depreciation and that, since capital and output are the same commodity, one can consume one's capital.)

Individuals in this economy live for two periods, working in the first while being retired in the second. Each person has an ordinal utility function $U(e^1, e^2)$ based on his consumption in the two years of his life.[2] Denoting the number of persons born at the start of the tth period by L_t, labor force, growth satisfies:

$$L_t = L_0(1 + n)^t.$$

2. Centrally Planned Economy

It is simplest to examine the production possibilities of this economy by examining the alternatives available to a central planning authority. With the capital stock in period t (which was determined in period $t-1$) and the labor force in this period (which is exogenous), output will satisfy $Y_t = F(K_t, L_t)$. At the end of the production process (and before the start of consumption in this period) the central authorities have command over the capital stock and the newly produced output, $K_t + Y_t$. This must be divided between the capital stock which will be available for production in the next period, K_{t+1}, and aggregate consumption in this period, C_t. This consumption must be further divided between members of the younger generation, E_t^1, and those of the older generation, E_t^2. Assuming that all members of the same generation consume the same amount, we have:[3]

$$\overset{1}{E_t} = \overset{1}{e_t} L_t, \; \overset{2}{E_t} = \overset{2}{e_t} L_{t-1}.$$

The division of the resources on hand between the alternative uses can be stated algebraically:

$$Y_t + K_t = K_{t+1} + C_t = K_{t+1} + \overset{1}{e_t} L_t + \overset{2}{e_t} L_{t-1}, \tag{1}$$

[1] It is assumed that F is twice differentiable and exhibits positive marginal products and a diminishing marginal rate of substitution everywhere.

[2] The assumption of the absence of all bequests is important for intertemporal allocation conclusions. A relationship between changes in the size of the national debt and changes in bequests can alter the effects to be described.

[3] Note that a person born in period t would consume e_t^1 and e^2_{t+1} in his two years of life.

or more conventionally,

$$Y_t - (K_{t+1} - K_t) = C_t = \overset{1}{e}_t L_t + \overset{2}{e}_t L_{t-1}. \tag{2}$$

Assuming that the central authorities decide to preserve a constant capital-labor ratio, $k_t = K_t/L_t$, and thus $K_{t+1} = (1+n)K_t$, aggregate consumption will satisfy:

$$Y_t - nK_t = C_t = \overset{1}{e}_t L_t + \overset{2}{e}_t L_{t-1}. \tag{3}$$

Denoting the output-labor ratio by $y_t = Y_t/L_t$, this can be rewritten as:

$$y_t - nk_t = C_t/L_t = \overset{1}{e}_t + \overset{2}{e}_t/(1 + n). \tag{4}$$

Maintenance of a constant capital-labor ratio implies, of course, a constant output per worker over time. Thus, this equation describes the consumption possibilities in each year of any period during which the capital-labor ratio remains constant. In particular, if a given capital-labor ratio is held constant for all time, the economy is on what has become known as a Golden Age Path.

3. *Neoclassical Stationary States or Golden Age Paths*

A Golden Age Path for an economy is an expansion path on which the capital-labor ratio (and thus the capital-output ratio and marginal product of capital) is kept constant. From equation (4) we see that the central-planning authorities can maintain any capital-labor ratio for which the output-capital ratio is not smaller than n (which is equivalent to the condition that the savings rate not exceed one). From equation (4), again, one can derive the amount of consumption that is possible in each period and thus calculate the Golden Age Path for which this is maximized. Similarly we can examine the alternative divisions of this consumption between individuals of different generations. Assuming that all individuals have the same lifetime consumption pattern, the problem of selecting the optimal Golden Age Path, the Golden Age Path on which each individual would have the highest utility level, subject to the constraint that all individuals have the same level, can be written:

$$\text{Maximize } U(e^1, e^2) \qquad \text{subject to } e^1 + e^2/(1 + n) = y - nk. \tag{5}$$

Thus the solution of the problem of selecting an optimal Golden Age Path treats the allocation of consumption over the lifetime of an individual in a similar fashion to the allocation of consumption, in a single year, between individuals of different ages. The selection between Golden Age Paths is, as is seen from (5), a selection which ignores initial conditions, and thus not a selection available to an economy, which must weigh the advantages of a given long-run equilibrium against the costs of achieving it.

4. *The Golden Rule Path*

This maximization problem decomposes naturally into two separate problems, that of selecting the optimal capital-labor ratio, and thus the height of the consumption constraint, $y-nk$; and that of dividing this amount of consumption between the different individuals. The maximizing capital-labor ratio is seen from (5) to satisfy the condition that the marginal product of capital equal the rate of growth, $F_K=n$. This is the standard result on the nature of the Golden Rule Path, see, e.g., Phelps [6]. Note that the optimality of this capital-labor ratio is independent of the exact division of consumption (and selecting the optimal division is independent of the capital-labor ratio chosen). If the central planners choose a higher capital-labor ratio, they would be selecting an inefficient solution (in the standard sense including the problem of initial conditions, not just as a comparison of Golden Age Paths) in that they could discard capital, lowering the capital-labor ratio to the Golden Rule level, and preserve this capital-labor ratio forever, permitting a higher level of consumption in each period forever.[4]

Utility maximizing consumption allocation clearly requires that

$$\frac{\partial U}{\partial e^1} = (1 + n) \frac{\partial U}{\partial e^2} \cdot$$

This is the allocation that would occur if consumption decisions were individually made employing a rate of interest for consumer decisions equal to the rate of growth. In examining the division of consumption when the capital-labor ratio is held constant, we are equivalently examining a model in which there is only one factor of production, labor. Thus it is not surprising that the optimal allocation is the same as that found by Samuelson [8], which he called the biological optimum. Thus the optimal rate of interest is determined by the rate of population growth (which may or may not equal the marginal product of capital). This paradox arises from the comparison of stationary states. The shifting of one unit of consumption by an individual from his first to his second year is equivalent to removing one unit of consumption from each of the living members of the younger generation and giving this total to the contemporary older generation, of whom there are n per cent fewer members.

[4] Dynamic inefficiencies of this sort in models both with and without technical change are examined by Phelps [7].

5. *Competitive Framework*

To the technological possibilities which have been described above, it is necessary to replace the central planning framework by a market process for the determination of the saving rate in each period. The annual savings behavior of the economy will determine the long-run equilibrium to which the economy converges. In particular, we will be interested in comparing alternative Golden Age Paths to which the economy converges with different quantitites of government debt outstanding. Thus, only the long-run implications of national debt will be examined, thereby avoiding the problem of selecting a social welfare function for the evaluation of different individual utility levels (at the cost of failing to explore the total effects).

By following the life history of a single individual, born, say, in period t, it is possible to trace out the market relations. This individual works in period t, for which he receives a wage, w_t, which equals the marginal product of labor, $F_L(K_t, L_t)$. This wage he allocates between current and future consumption so as to maximize his utility function, given the rate of interest existing on one-period loans from period t to period $t+1$, r_{t+1}. Thus, the members of the younger generation make up the supply side of the capital market.

This individual will thus consume, in period t, the difference between his wage and the quantity he lends in the capital market, $e^1{}_t = w_t - s_t$. In period $t+1$, he will consume his savings plus the accrued interest, $e^2{}_{t+1} = (1+r_{t+1})s_t$.

Capital demanders are entrepreneurs who wish to employ capital for production in period $t+1$. Thus the equilibrium interest rate will equal the marginal product of capital, $r_{t+1} = F_K(K_{t+1}, L_{t+1})$.

6. *Factor-Price Frontier*

The existence of the constant returns to scale production function, $F(K, L)$, which can be written as $Lf(k)$, implies a relationship between the marginal products of labor and capital which will be denoted by $w = \phi(r)$.[5] From the definitions $r = f'(k)$ and $w = f(k) - kf'(k)$ we see that:

$$(6) \qquad \frac{dw}{dr} = \phi'(r) = -k \quad \text{and} \quad \frac{d^2w}{dr^2} = \phi''(r) = \frac{-1}{f''(k)} \cdot$$

7. *Utility Maximization*

Utility maximization,[6] given a wage level and a market interest rate,

[5] For a description of the factor-price frontier see Samuelson [9].

[6] It is assumed that the utility function has the following properties: no satiation, a diminishing marginal rate of substitution everywhere, and a shape which guarantees that consump-

implies that consumption will be allocated so that:

$$\frac{\partial U}{\partial e^1} = (1 + r) \frac{\partial U}{\partial e^2} \cdot$$

Therefore, the quantity saved can be expressed as a function of the relevant wage and interest level, $s_t = s(w_t, r_{t+1})$. It will be assumed that s is a differentiable function. From the assumption of normality, we have $0 < \partial s/\partial w < 1$. However $\partial s/\partial r$ may be positive or negative.

In addition to writing individual savings as a function of the wage and interest rates, it is possible to express the utility function in terms of these variables. From this derived form of the utility function one has:[7]

$$(7) \qquad \frac{\partial U}{\partial w} = \frac{\partial U}{\partial e^1}, \qquad \frac{\partial U}{\partial r} = \frac{s}{(1 + r)} \frac{\partial U}{\partial e^1} \cdot$$

8. Capital Market

From the discussion above, we know that we can write the supply schedule of capital, which is the sum of the individual savings functions, as:

$$(8) \qquad S_t = s_t L_t = L_t s(w_t, r_{t+1}).$$

The demand curve for capital, which relates the capital stock in period $t+1$ to the interest rate, is merely the marginal product of capital as a function of the capital-labor ratio:

$$(9) \qquad r_{t+1} = f'(K_{t+1}/L_{t+1}).$$

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tion in each period is a normal good, i.e.,

$$0 < \frac{\partial s}{\partial w} < 1.$$

[7] Using the optimality condition, we have:

$$\frac{\partial U}{\partial w} = \frac{\partial U}{\partial e^1} \frac{\partial e^1}{\partial w} + \frac{\partial U}{\partial e^2} \frac{\partial e^2}{\partial w} = \frac{\partial U}{\partial e^1} \left[\frac{\partial e^1}{\partial w} + \left(\frac{1}{1 + r} \right) \frac{\partial e^2}{\partial w} \right].$$

From the net worth constraint $e^1 + e^2/(1+r) = w$, we have

$$\frac{\partial e^1}{\partial w} + \frac{\partial e^2}{\partial w}/(1 + r) = 1,$$

which, upon substitution, yields equation (7). Similarly, the net worth constraint implies that

$$\frac{\partial e^1}{\partial r} + \frac{\partial e^2}{\partial r}/(1 + r) - \frac{e^2}{(1 + r)^2} = 0.$$

Thus

$$\frac{\partial U}{\partial r} = \frac{\partial U}{\partial e^1} \left[\frac{\partial e^1}{\partial r} + \left(\frac{1}{1 + r} \right) \frac{\partial e^2}{\partial r} \right] = \frac{\partial U}{\partial e^1} \frac{e^2}{(1 + r)^2} = \frac{s}{(1 + r)} \frac{\partial U}{\partial e^1} \cdot$$

Combining the demand and supply curves, equating S_t and K_{t+1}, we have the equilibrium condition in the capital market, which relates the interest rate to the wage rate of the previous period:

$$r_{t+1} = f'(S_t/L_{t+1}) = f'(s(w_t, r_{t+1})/(1 + n)). \tag{10}$$

From the assumptions made above, we know that the demand curve is downward-sloping, while the supply curve may have a positive or negative slope. This suggests that there are two cases which need to be treated separately as the demand or supply curve is more steeply negatively sloped.[8] This is shown diagrammatically in Diagram 1.

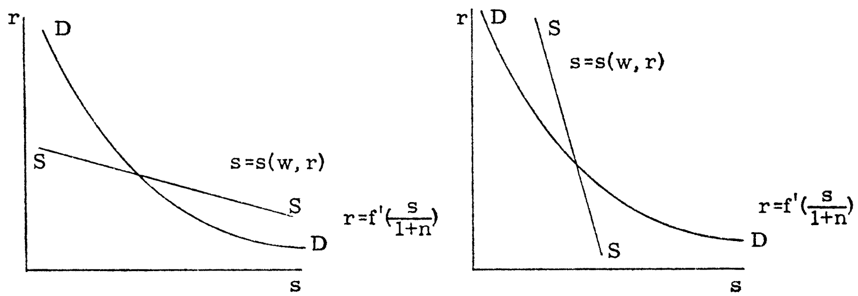

Diagram 1

The necessity of distinguishing the two cases is made clear by examining the relation between the equilibrium interest rate and the wage of the previous period. A higher wage in period t implies a greater quantity of saving at any interest rate, or a rightward shift of the saving curve in Diagram 1. However, whether this results in a higher or lower equilibrium level of saving depends on the relative slopes of the demand and supply curves. Geometrically, assuming $w' > w$, we have Diagram 2.

In the diagram on the right, which represents the "normal" case in the capital market, a higher income level results in a higher equilibrium level of saving. In the diagram on the left, where the elasticity of saving with respect to the interest rate is large and negative, a rise in the level of income results in a fall in the equilibrium level of saving. This somewhat perverse case leads to a reversal of the conclusions on the effect of debt (since taxes which reduce disposable income increase saving). Rather than complicate the text, we discuss this case in Appendix A.

By altering the wage levels in period t, we could trace out the equilibrium interest rates which would occur in period $t+1$. This relation

[8] A requirement of Walrasian stability in the capital market would permit an elimination of the case where the supply curve is steeper than the demand curve. Marshallian stability would not, of course, permit this elimination. In the absence of a dynamic theory of the capital market, it seems best to consider both cases.

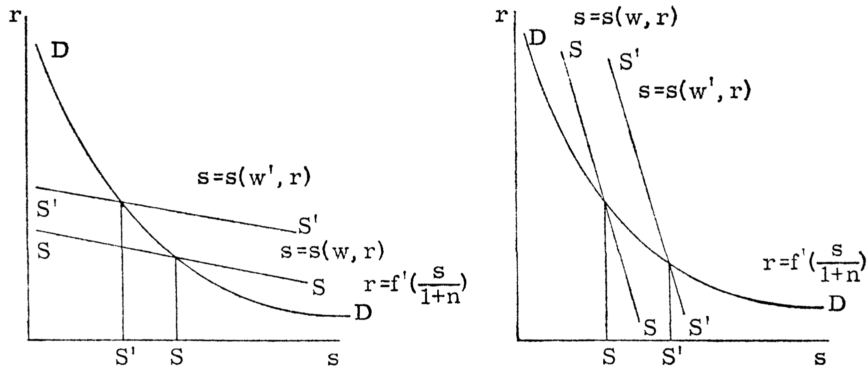

Diagram 2

will be denoted by $r_{t+1}=\psi(w_t)$. It will be assumed that ψ is differentiable. From the assumption on the relative slopes of the demand and supply curves for capital, we know that an increase in wages implies an increase in saving and thus a decrease in the interest rate. Taking the derivative of r with respect to w, we can express this as:

$$\frac{dr_{t+1}}{dw_t} = \psi' = \frac{f''\dfrac{\partial s}{\partial w}}{1+n-f''\dfrac{\partial s}{\partial r}} < 0. \tag{11}$$

9. Competitive Solution

The history of this economy can be traced in Diagram 3 containing the ψ function (relating r_{t+1} to w_t) and the ϕ function (relating w_t and r_t).

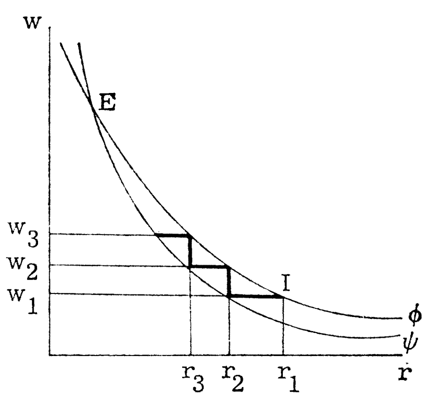

Diagram 3

Given a wage and interest pair in period one, (w_1, r_1), which is denoted by I in Diagram 3, the interest rate in the second period can be read from the ψ curve, given the wage in period one. With this interest rate in period two, the factor-price frontier, ϕ, gives the value of the wage in period two. The entire time path of the economy can be traced out in this diagram in similar fashion.

As portrayed in Diagram 3, and as will be assumed throughout this paper, the economy has a single, stable equilibrium point. In order to derive this stability condition (which will be used to derive the direction of changes in equilibrium values when debt is introduced), one first expresses r_{t+1} as a function of r_t: $r_{t+1}=\psi(\phi(r_t))$. Taking the derivative of this, and recalling equation (11) which implies that this derivative is positive, we can express the necessary condition for stability as:

$$0 < \frac{dr_{t+1}}{dr_t} = \psi'\phi' = \frac{-kf''\dfrac{\partial s}{\partial w}}{1+n-f''\dfrac{\partial s}{\partial r}} \leqq 1. \tag{12}$$

As is shown by the example in the next section, the competitive solution need not occur at an interest rate exceeding the Golden Rule level. Thus the competitive solution may be dynamically inefficient[9] since there exists a time after which the capital-labor ratio will exceed the Golden Rule level by a nonvanishing amount.[10]

10. An Example

As an example, consider an economy with Cobb-Douglas production and utility functions. The utility function can be expressed as:

$$U(e^1, e^2) = \beta \log e^1 + (1-\beta) \log e^2.$$

The saving function derived from this is independent of r:

$$s = (1-\beta)w.$$

Thus ψ can be written:

$$r_{t+1} = f'\left(\frac{(1-\beta)w_t}{(1+n)}\right).$$

With production satisfying:

$$y = Ak^\alpha,$$

[9] The possibility of an inefficient solution in an economy with infinitely many decision makers has been discussed by Koopmans [2].

[10] That this implies dynamic inefficiency is proved by Phelps [7].

ψ becomes:

$$r_{t+1} = \alpha A \left(\frac{(1-\beta)w_t}{(1+n)}\right)^{\alpha-1},$$

while ϕ can be written:

$$w_t = (1-\alpha)\alpha^{\alpha/1-\alpha}A^{1/1-\alpha}r_t^{\alpha/\alpha-1}.$$

Combining these we have:

$$r_{t+1} = \left(\frac{\alpha(1+n)}{(1-\beta)(1-\alpha)}\right)^{1-\alpha} r_t^{\alpha}.$$

The long-run equilibrium thus satisfies:

$$r^E = \lim_{t\to\infty} r_t = \frac{\alpha(1+n)}{(1-\alpha)(1-\beta)}.$$

Except if

$$n = \frac{\alpha}{(1-\alpha)(1-\beta)-\alpha},$$

this does not coincide with the Golden Rule. With a positive rate of growth of labor, different economies with different values of α or β can clearly have interest rates either larger or smaller than n.

11. Framework of Analysis

In examining the long-run effects of national debt, there are two approaches that might be taken, corresponding to the two concepts of incidence, balanced-budget incidence and differential incidence.[11] With balanced-budget incidence, the effects of a combination of changed expenditures and changed financing are examined, weighing the relative benefits and costs. Differential incidence refers to a comparison of alternative methods of financing a given expenditure level.[12]

In this model, there are two forms which government expenditures could take, a current consumption item (which might best be viewed as lump-sum gifts to part of the populace) or government purchase of physical capital (which would then be rented to entrepreneurs for use in the production process in each future period, with the rental payments distributed to the individuals as a social dividend). Combining these two forms of expenditures with the alternatives of tax or debt

[11] For a detailed discussion of these concepts, see Musgrave [5].

[12] The failure to distinguish between these separate questions has been the cause of some of the confusion in the literature on the public debt. See, e.g., Mishan [3].

finance gives four possible questions of balanced-budget incidence which might be asked.

However, although answers to some of these questions will arise, analysis will be restricted to the differential incidence question of substituting debt for tax finance for a given government expenditure.

This substitution could be employed while financing the purchase of physical capital. The long-run incidence question would then be resolved by comparing the long-run equilibrium arising when there is government capital and government debt with the long-run equilibrium occurring when there is only government capital. Since the simultaneous issuance of debt and purchase of capital would merely make the government a middleman between entrepreneurs and savers, this action would have no effect on the economy in either the short or long run. Thus the initial equilibrium would be compared to that arising when there is government-owned capital, but no debt outstanding.

Alternatively, the government could finance some windfall payment (such as veterans' bonuses) either from concurrent taxes or debt issuance. While tax-financed transfer payments would have an effect in the short run (depending on the relation between the recipients and the taxpayers), since it shifts neither ϕ nor ψ it would have no effect on the long-run equilibrium. Thus the original long-run equilibrium could be compared to the one arising when debt exists (which shifts ψ) but the expenditures had no permanent effect. Either of these differential incidence frameworks would lead to the same qualitative solutions, and the second one will be adopted.

12. National Debt

To avoid the problem of expected capital gains, it will be assumed that all government debt has a one-period maturity. It will also be assumed that the debt, which is refloated each period simultaneously with the achievement of equilibrium in the capital market, pays the current interest rate. For internally held debt this assumption is necessary, given the assumption of perfect certainty, for wealth owners to be willing to hold both debt and physical capital in their portfolios. The assumption is also made for externally held debt for the sake of symmetry in the comparison of the two types of debt. The, perhaps, more natural assumption of a supply curve of external capital is discussed in Appendix B. With the assumption of a horizontal supply curve at an interest rate equalling the equilibrium domestic rate before the issuance of further debt, the qualitative results of this case are identical to those of the case considered in the text.

Since the long-run effects of the debt depend on a permanent shift in ϕ or ψ, a fixed absolute amount of debt, in a growing economy, would asymptotically have no effect. Therefore it will be assumed that the debt-labor ratio is held constant (that the quantity of debt grows at n per cent) by financing part of the interest cost by additional debt, while financing the remainder by taxes. (It should be noted that in the case of an inefficient competitive solution, where the rate of growth exceeds the rate of interest, this implies negative taxes.) The measure of the quantity of debt outstanding in any period will be the quantity outstanding at the start of the period (or equivalently, at the time of the production process), which is therefore the quantity issued in the previous period. Thus the denominator of the debt-labor ratio refers to the number of individuals in the tax base for financing the debt, rather than the number of savers entering the capital market to purchase the debt.

The taxes employed to finance interest costs (which are paid concurrently with the receipt of factor payments) will be assumed to be lump-sum taxes on the younger generation.[13]

13. External Debt

The effects of the existence of externally held debt on the domestic economy arise solely from the taxes needed to finance that part of the interest cost not covered by increased debt. Thus we would expect the utility of an individual living at the time of long-run equilibrium to decrease because of increased taxes (in the efficient case where the interest rate exceeds the growth rate) and to change because of the change in the equilibrium wage-interest rate pair caused by the impact of these taxes on the supply side of the capital market. Denoting the external debt-labor ratio by g_1, the taxes per worker in period t are $(r_t - n)g_1$. Therefore, the equation for the ψ function must be changed to relate savings to the wage net of taxes $\hat{w}_t$, which equals $w_t - (r_t - n)g_1$. Rewriting the condition for equilibrium in the capital market, equation (10), we have:

$$r_{t+1} = f'\left(\frac{s(w_t - (r_t - n)g_1, r_{t+1})}{1 + n}\right). \tag{13}$$

The new form of this equation implies a new stability condition which, together with the assumption on relative slopes in the capital market, is expressed in (14). (It is assumed that there is a single stable equilibrium both with and without the debt.)

[13] The case with lump-sum taxes on the older generation is equivalent to that with lump-sum taxes on the younger generation plus intergeneration transfers in each period. This transfer scheme (from old to young) increases saving in anticipation of taxes and counteracts the decreases described in the text, although not fully.

$$(14) \qquad 0 < \frac{dr_{t+1}}{dr_t} = \frac{-f''(k+g_1)\dfrac{\partial s}{\partial w}}{1+n-f''\dfrac{\partial s}{\partial r}} \leqq 1.$$

To examine the shift in the ψ curve, we can implicitly differentiate equation (13) again, this time taking the partial derivative of r_{t+1} with respect to g_1.

$$(15) \qquad \frac{\partial r_{t+1}}{\partial g_1} = \frac{f''(n-r_t)\dfrac{\partial s}{\partial w}}{1+n-f''\dfrac{\partial s}{\partial r}}.$$

From equation (14) we know that the sign of this expression is the same as that of $(r-n)$. Geometrically we have ψ shifting to the new curve ψ' as shown in Diagram 4. By combining the new ψ curve with

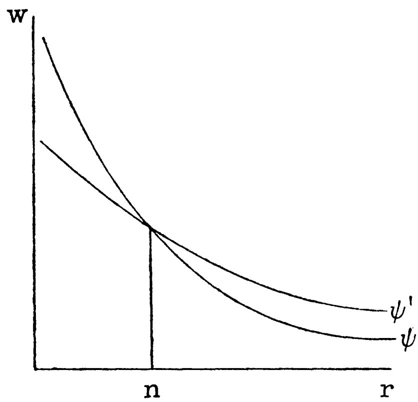

Diagram 4

the factor price frontier, which is unchanged, we can examine the change in the long-run equilibrium values of r and w. In Diagram 5 we see that, if the equilibrium interest rate was unequal to the growth rate, the existence of external debt increases the difference between the two.[14]

To examine the effects of external debt on utility levels in long-run equilibria, it is simplest to assume a given level of external debt and examine the changes arising from a derivative change in this quantity.

[14] Since, if $r=n$, additional debt issuance exactly covers interest payments, if this were the original equilibrium, the debt has no effect.

Using equation (13) and the constancy of the interest rate in long-run equilibrium, one can express the equilibrium interest rate as an implicit function of the quantity of debt outstanding:

$$r = f'\left(\frac{s(\phi(r) - (r - n)g_1, r)}{1 + n}\right). \tag{16}$$

From this relationship one can derive the change in the equilibrium interest rate arising from a change in the debt-labor ratio:

$$\frac{dr}{dg_1} = \frac{-f''(r - n)\dfrac{\partial s}{\partial w}}{1 + n - f''\dfrac{\partial s}{\partial r} + f''(k + g_1)\dfrac{\partial s}{\partial w}}. \tag{17}$$

As was described above, external debt moves the interest rate away from the Golden Rule solution. In terms of the capital market, we have that positive taxes, by decreasing the supply of capital, given any level of the wage, increase the equilibrium interest rate.

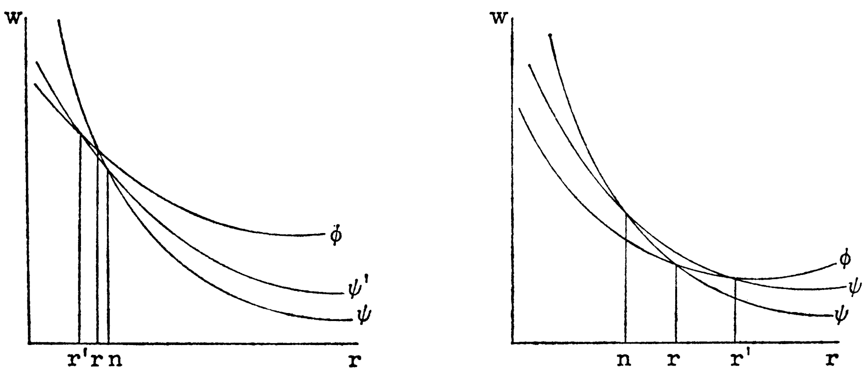

Diagram 5

The change in the utility level can be calculated by employing the expressions for the effects of changes in factor payments on the utility level, equation (7).

$$\frac{dU}{dg_1} = \frac{d\hat{w}}{dg_1}\frac{\partial U}{\partial w} + \frac{dr}{dg_1}\frac{\partial U}{\partial r} = \frac{\partial U}{\partial e^1}\left[\frac{d\hat{w}}{dg_1} + \frac{s}{(1 + r)}\frac{dr}{dg_1}\right]. \tag{18}$$

From the expression for the net wage, $\hat{w} = w - (r - n)g_1$, one can calculate the change in the net wage in terms of the change in the interest rate:

$$\frac{d\hat{w}}{dg_1} = -(k + g_1)\frac{dr}{dg_1} - (r - n). \tag{19}$$

Substituting this equation in the previous equation we have:

$$\frac{dU}{dg_1} = -\frac{\partial U}{\partial e^1}\left[(r-n) + g_1\frac{dr}{dg_1} + \left(k - \frac{s}{1+r}\right)\frac{dr}{dg_1}\right]. \tag{20}$$

The first term of this expression is the change in utility arising from the taxes needed to finance the addition to the outstanding debt, and is positive or negative as these taxes are positive or negative. The second term describes the change in the tax burden of existing debt occurring because of the change in the interest rate. Thus, both of these utility changes are positive if r exceeds n and negative if n exceeds r.

The third term can be explained by means of Diagram 6 containing derived indifference curves between w and r, denoted by II, and the factor-price frontier.[15]

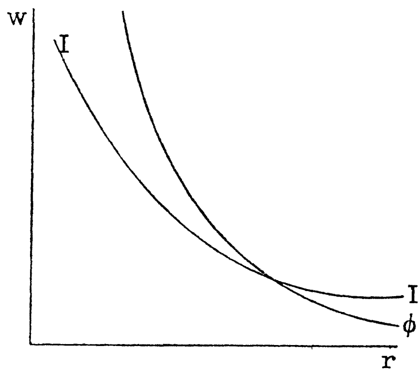

Diagram 6

The change in the interest rate is a movement along the factor-price frontier. The change in utility thus depends on the relative slopes of the factor-price frontier, the slope of which is $-k$, and of the indifference curve, the slope of which is $-s/(1+r)$. From the equilibrium condition for the capital market, $k=s/(1+n)$, this term can be rewritten as

$$\frac{dr}{dg_1}\left(\frac{k}{1+r}\right)(r-n).$$

Since dr/dg_1 has the same sign as $(r-n)$, the movement of the interest rate away from the Golden Rule level causes the utility from factor payments to fall.

Combining the three effects, we can conclude that in the "normal" case where the competitive solution is efficient, external debt causes a

[15] A rigorous treatment of this approach would require changes in the indifference levels because of taxes.

fall in the utility level of an individual living in long-run equilibrium. If the competitive solution is inefficient the effects of the debt work in opposite directions and so yield no a priori conclusion.

14. *Internal Debt*

With internal debt, the supply side of the capital market is altered in precisely the same fashion as with external debt, since an individual taxpayer is in the same position as a taxpayer whether his tax payments flow abroad or remain in the country. Denoting the internal debt-labor ratio by g_2, the savings function must be altered as before to read

$$s(w_t - (r_t - n)g_2, r_{t+1}).$$

It is also necessary to alter the equilibrium condition in the capital market to take account of the fact that the government enters on the demand side of this market. Denoting the quantity of internal debt to be floated in period t (and repaid in $t+1$) by G_{t+1}, the equilibrium condition becomes:

$$S_t = K_{t+1} + G_{t+1}. \tag{21}$$

Dividing this by L_{t+1}, we have the equilibrium condition expressed in terms of the ratios needed to describe the equilibrium:

$$\frac{s_t}{1+n} = k_{t+1} + g_2. \tag{22}$$

Comparing internal with external debt, we see that they both require taxes to be paid by each worker, while internal debt has a further effect in that it substitutes pieces of paper for physical capital in the portfolios of wealth owners, thus reducing output.

Recalling that the demand for capital by entrepreneurs is determined by the marginal productivity schedule of capital, we can combine this equation with equation (22) to obtain the new condition for equilibrium in the capital market:

$$r_{t+1} = f'\left(\frac{s(w_t - (r_t - n)g_2, r_{t+1})}{1+n} - g_2\right). \tag{23}$$

As before, by implicit differentiation of this equation we can express the conditions for stability and the sssumed slopes in the capital market:

$$0 < \frac{dr_{t+1}}{dr_t} = \frac{-f''(k + g_2)\dfrac{\partial s}{\partial w}}{1 + n - f''\dfrac{\partial s}{\partial r}} \leqq 1. \tag{24}$$

To find the shift in ψ, we take the partial derivative of r with respect to g_2:

$$\frac{\partial r_{t+1}}{\partial g_2} = \frac{-f''\left(\frac{\partial s}{\partial w}(r-n)+(1+n)\right)}{1+n-f''\frac{\partial s}{\partial r}}. \tag{25}$$

From (24) and the normality of present consumption, $\partial s/\partial w<1$, we know that this expression is positive and thus that ψ shifts upward for all values of r.

Following the same analysis as with external debt, we can calculate the change in utility arising from the change in the level of internal debt. We write first the locus of equilibria for different quantities of debt:

$$r = f'\left(\frac{s(\phi(r)-(r-n)g_2, r)}{1+n} - g_2\right). \tag{26}$$

We can then differentiate this expression with respect to g_2 to obtain the change in the equilibrium interest rate arising from the change in the quantity of debt:

$$\frac{dr}{dg_2} = \frac{-f''\left(1+n+(r-n)\frac{\partial s}{\partial w}\right)}{1+n-f''\frac{\partial s}{\partial r}+f''(k+g_2)\frac{\partial s}{\partial w}}. \tag{27}$$

Again, from equation (24), we know that the change in the equilibrium interest rate is positive. Employing equations (18) and (19) relating changes in utility and the net wage to changes in debt (which hold for either internal or external debt) and the equilibrium condition for the capital market, equation (22), we can express the changes in utility in two ways:

$$\frac{dU}{dg_2} = -\frac{\partial U}{\partial e^1}\left[(r-n)+g_2\frac{dr}{dg_2}+\left(k-\frac{s}{1+r}\right)\frac{dr}{dg_2}\right] \tag{28}$$

$$\frac{dU}{dg_2} = -\frac{\partial U}{\partial e^1}(r-n)\left[1+\frac{k+g_2}{1+r}\frac{dr}{dg_2}\right]. \tag{29}$$

Equation (28) expresses the change in utility in terms of the taxes needed to finance the increase in debt, the taxes needed to finance the increased interest payments on existing debt, and the changed value of factor payments. As before, the sign of the first two terms depends solely on $(r-n)$. However, since $s=(1+n)\ (k+g_2)$, the third term, while decreasing utility when r is smaller than n, may increase or decrease utility when the competitive solution is efficient.

Equation (29), which combines the separate terms, shows that utility is decreased in the efficient case and increased in the inefficient case. Like the third term in the expression giving the change in utility from external debt, the sign of this expression is the opposite of the sign of $(r-n)dr/dg$. Separating the effects of debt issuance into those which alter the social consumption possibilities (the flow of interest payments abroad) and those that reflect a change in the allocation of consumption within the society, the total impact of all effects falling into the second group will increase or decrease utility as the interest rate is moved toward or away from the rate of growth. This is demonstrated geometrically in the next section.

15. Diagrammatic Discussion

The assumption that both present and future consumption are normal goods implies that, as one moves upward and to the left along a budget line, the indifference curves become steeper. Geometrically, this implies that the slope at A is algebraically greater than at B. See Diagram 7.

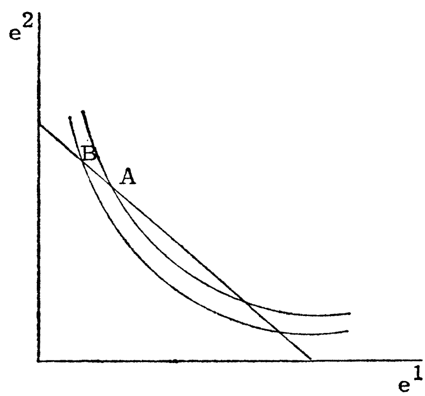

Diagram 7

Recalling equation (5), the consumption possibilities for a society with a given capital-labor ratio lie along the line $e^1+e^2/(1+n)=y-nk$. Since the interest rate is the marginal product of capital in a competitive society, from the interest rate we know the height of the consumption constraint line. Furthermore, since consumption is allocated over time in accordance with the market interest rate, we know that the competitive equilibrium occurs where the slope of an indifference curve equals $-(1+r)$. These two facts permit us to locate a competitive equilibrium in Diagram 7, knowing just the equilibrium interest rate (and, of course, the production function). Since internal debt does not alter the consumption possibilities available to an economy, the utility associated with the equilibrium arising from varying quantities of internal debt can be located in this diagram. (This, of course, is not true for external debt.)

Combining these two facets of a change in the interest rate we can conclude that any movement of the interest rate away from the growth rate decreases utility first by diminishing the height of the consumption constraint line and second by moving along the lowered line in the direction of decreased utility. Assuming $r'>r>n$, this is shown in Diagram 8 where A is the equilibrium point associated with r while C is the one associated with r'.

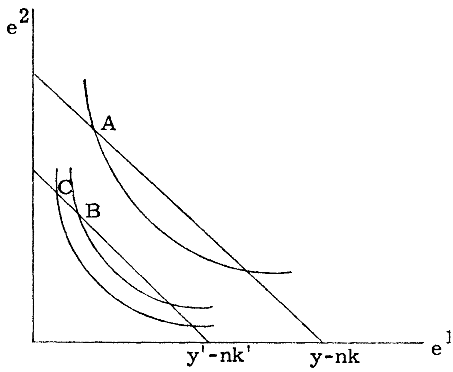

Diagram 8

Utility at B (where the slope of the indifference curve is the same as at A) is less than utility at A since B is on a lower constraint line. The slope at C, which equals $-(1+r')$, is less than at B, $-(1+r)$, implying a lower utility level at C than at B.

Thus internal debt raises or lowers the utility level as it moves the equilibrium interest rate towards or away from the growth rate. External debt has two effects, an alternation of consumption possibilities due to the flow abroad of interest payments and an alteration of utility arising from changes in the interest rate, given the level of interest payments, which is positive or negative as the interest rate is moved toward or away from the growth rate.

Thus the third term in dU/dg_1, which is the "purely domestic" effect of issuing external debt corresponds to the entire effects of internal debt, with the sign of the expressions equalling that of $-(r-n)dr/dg$.

16. Internal and External Debt

Having described the way each of them affects the equilibrium of the economy, it is now possible to turn to the complete model, in which there is both external and internal debt, and so make a direct comparison of their effects.[16]

Without stopping to repeat the analysis step by step, we can write down the relevant equations from the equations derived in the last three sections:

The condition for equilibrium in the capital market:

$$r_{t+1} = f'\left(\frac{s(w_t - (r_t - n)(g_1 + g_2), r_{t+1})}{1 + n} - g_2\right). \tag{30}$$

The locus of long-run competitive equilibria with different quantities of debt outstanding:

$$r = f'\left(\frac{s(\phi(r) - (r - n)(g_1 + g_2), r)}{1 + n} - g_2\right). \tag{31}$$

The necessary condition for stability and the assumption on the demand and supply curves for capital:

$$0 < \frac{dr_{t+1}}{dr_t} = \frac{-f''(k + g_1 + g_2)\dfrac{\partial s}{\partial w}}{1 + n - f''\dfrac{\partial s}{\partial r}} \leqq 1. \tag{32}$$

The change in the equilibrium interest rate arising from changes in debt:

$$\begin{aligned} \frac{dr}{dg_1} &= \frac{-f''(r - n)\dfrac{\partial s}{\partial w}}{1 + n - f''\dfrac{\partial s}{\partial r} + f''(k + g_1 + g_2)\dfrac{\partial s}{\partial w}} \\ \frac{dr}{dg_2} &= \frac{-f''\left((r - n)\dfrac{\partial s}{\partial w} + (1 + n)\right)}{1 + n - f''\dfrac{\partial s}{\partial r} + f''(k + g_1 + g_2)\dfrac{\partial s}{\partial w}}. \end{aligned} \tag{33}$$

[16] In Section 14, the change from external to internal debt was described as adding the effect arising from the substitution of paper for physical capital in portfolios. Reversing this comparison, external debt is internal debt plus annual foreign borrowing, with foreign capital receiving its marginal product. This does not directly alter net output, but it does alter relative factor prices which directly affects utility and which alters savings.

The changes in utility arising from changes in the quantity of debt:

$$(34)\qquad \begin{aligned} \frac{dU}{dg_1} &= -\frac{\partial U}{\partial e^1}\left((r-n)+\frac{dr}{dg_1}(g_1+g_2)+\frac{dr}{dg_1}(k-\frac{s}{1+r})\right) \\ \frac{dU}{dg_2} &= -\frac{\partial U}{\partial e^1}\left((r-n)+\frac{dr}{dg_2}(g_1+g_2)+\frac{dr}{dg_2}(k-\frac{s}{1+r})\right). \end{aligned}$$

With these relations before us, it is possible to examine the differential incidence question arising from the effects of issuing internal debt to retire external debt, and to examine the relationships between some of the articles in the literature on the burden of the debt.

From equation (33) we can calculate the change in the equilibrium interest rate arising from this debt swap:

$$(35)\qquad \frac{dr}{dg_2}-\frac{dr}{dg_1}=\frac{-f''(1+n)}{1+n-f''\dfrac{\partial s}{\partial r}+f''(k+g_1+g_2)\dfrac{\partial s}{\partial w}}.$$

From the stability condition, (32), we know that the denominator of this expression is positive and thus that the interest rate always rises. The effect of the debt swap involves no change in taxes, and so no change in the supply side of the capital market. However, the demand side is altered by the increase in government demand, causing a rise in the equilibrium interest rate and a fall in the capital-labor ratio.

The change in utility can be derived from (34) and expressed in different ways:

$$(36)\qquad \frac{dU}{dg_2}-\frac{dU}{dg_1}=-\frac{\partial U}{\partial e^1}\left[\frac{dr}{dg_2}-\frac{dr}{dg_1}\right]\left[(g_1+g_2)+\left(k-\frac{s}{1+r}\right)\right],$$

$$(37)\qquad \begin{aligned} \frac{dU}{dg_2}-\frac{dU}{dg_1} = -\frac{\partial U}{\partial e^1}&\left(\left[-(r-n)+\left(\frac{dr}{dg_2}-\frac{dr}{dg_1}\right)g_1\right]\right. \\ &\left.+(r-n)\left[1+\left(\frac{dr}{dg_2}-\frac{dr}{dg_1}\left(\frac{(k+g_2)}{(1+r)}\right]\right)\right), \end{aligned}$$

$$(38)\qquad \frac{dU}{dg_2}-\frac{dU}{dg_1}=-\frac{\partial U}{\partial e^1}\left[\frac{dr}{dg_2}-\frac{dr}{dg_1}\right]\left[\frac{(k+g_2)(r-n)}{(1+r)}+g_1\right].$$

Equation (36) divides the utility change into the part arising from the change in taxes and the part arising from the change in the utility of factor payments. Since the interest rate rises, taxes must rise, lowering utility. However, since

$$\left(k-\frac{s}{(1+r)}\right)\quad\text{is equal to}\quad\frac{(r-n)k-(1+n)g_2}{1+r},$$

as in the discussion of internal debt, the change in utility coming from the change in factor payments may be positive or negative.

Equation (37) divides the utility change into the part arising from the change in the external interest payments, which may be positive or negative, and the part arising, as in the last section, domestically, from the change in equilibrium values, given the level of external payments. This term has the sign

$$-(r-n)\left(\frac{dr}{dg_2}-\frac{dr}{dg_1}\right),$$

which is, therefore, the sign of $n-r$. Thus, as before, for this term the rise in interest rates raises utility in the inefficient case but lowers it in the efficient case.

The third form of the equation, (38), is expressed to most easily give the sign of the utility change. When the solution is efficient, we have an unambiguous fall in utility from this debt swap. In the inefficient case the sign depends on the relative sizes of

$$g_1, \quad \text{and} \quad \frac{(k+g_2)(r-n)}{(1+r)}\cdot$$

17. Conclusion

Thus we have seen that, where both types of debt exist, internal debt, which raises the interest rate, lowers utility in the efficient case but may raise or lower it in the inefficient case (if there were no external debt, utility would be raised in the inefficient case). External debt, which moves the interest rate away from the growth rate, lowers utility in the efficient case and may raise or lower it in the inefficient case (this remains true whether or not internal debt exists). Finally, the substitution of internal for external debt, which raises the interest rate, lowers utility in the efficient case, while being capable of raising or lowering it in the inefficient case.

There are two ways of classifying the effects of external and internal debt which shed some light on some of the effects described in the literature.

First, as in equation (34), they can be divided into utility changes arising from changes in taxes paid and from a change in the relative factor payments. This division shows that the taxes needed to finance either internal or external debt have the same impact on individuals living during long-run equilibrium.

Second, the change in utility from internal debt can be separated into the effects of external debt plus the effects of a debt swap. This would imply four effects, the effects of the two changes in taxes, and the two effects on factor payments. These latter two effects can be distinguished by the fact that external debt affects only the supply side of the capital market, while the debt swap affects only the demand side.

In their discussion of the effects of debt, Bowen, Davis, and Kopf [1] concentrated on the tax effects of internal debt, and so described the first two of these four effects.

Modigliani [4] and Vickrey [10] discussed the fall in the capital stock arising from the substitution of debt for capital in the portfolios of wealth owners. As such, they were discussing the change in the demand side of the capital market[17] and the effects described are additive to those arising from taxes. It is only necessary to add the effects of taxes on the capital stock (and thus on factor payments) to complete the discussion.

Appendix A

Making the alternative assumption on the capital market, which, together with the stability condition, can be expressed as:

$$-1 \leqq \frac{dr_{t+1}}{dr_t} < 0,$$

we can re-examine the signs of the equations in Section 16. The denominators of the expressions giving the change in the equilibrium interest rate are now negative (where they were positive in the text). Thus external debt moves the interest rate toward the growth rate, while internal debt lowers its equilibrium value. Consequently the debt swap lowers the rate of interest.

Therefore, in the efficient case, increased debt causes positive taxes for the additional debt but lowers the taxes on existing debt and so may raise or lower the utility level. The debt swap raises utility by decreasing taxes

[17] Modigliani described a one-for-one replacement of capital by debt, assuming that total wealth remained constant. However, the fall in the capital stock, which causes a fall in output, would affect the equilibrium quantity of total wealth. (Modigliani acknowledges this but ignores its effects.) The change in the capital stock can be derived from equation (33):

$$\frac{dk}{dg_1} - \frac{dk}{dg_2} = \frac{1}{f''}\left(\frac{dr}{dg_1} - \frac{dr}{dg_2}\right) = \frac{-1-n}{1+n-f''\left(\dfrac{\partial s}{\partial r} - (k+g_1+g_2)\dfrac{\partial s}{\partial w}\right)}.$$

This differs from -1 because of the term

$$\left(\frac{\partial s}{\partial r} - (k+g_1+g_2)\frac{\partial s}{\partial w}\right).$$

This latter expression represents the partial effect on desired wealth (which is equal to savings) arising from the fall in the capital stock:

$$\frac{\partial s}{\partial k} = \frac{\partial s}{\partial r}\frac{\partial r}{\partial k} + \frac{\partial s}{\partial w}\frac{\partial \hat{w}}{\partial k} = f''\frac{\partial s}{\partial r} + \frac{\partial s}{\partial w}f''\frac{\partial \hat{w}}{\partial r} = f''\left(\frac{\partial s}{\partial r} - (k+g_1+g_2)\frac{\partial s}{\partial w}\right).$$

and increasing the utility of factor payments by moving the interest rate toward the Golden Rule level.

Appendix B

Assuming a supply curve of external debt which can be expressed by writing the interest rate, ρ, as a function of g_1 (this assumes that the source of foreign capital is also increasing at n per cent per year), the net wage can be expressed:

$$\hat{w} = w - (\rho - n)g_1 - (r - n)g_2.$$

This implies that the change in the net wage can be expressed as:

$$\frac{d\hat{w}}{dg_1} = -(k + g_2)\frac{dr}{dg_1} - (\rho - n) - g_1\frac{d\rho}{dg_1},$$

$$\frac{d\hat{w}}{dg_2} = -(k + g_2)\frac{dr}{dg_2} - (r - n).$$

Thus the utility change becomes:

$$\frac{dU}{dg_1} = -\frac{\partial U}{\partial e^1}\left(\rho - n + g_1\frac{d\rho}{dg_1} + \frac{dr}{dg_1}\left(k + g_2 - \frac{s}{1 + r}\right)\right),$$

$$\frac{dU}{dg_2} = -\frac{\partial U}{\partial e^1}\left((r - n) + \frac{dr}{dg_2}\left(k + g_2 - \frac{s}{1 + r}\right)\right).$$

If the supply curve of capital is horizontal at the prevailing internal interest rate, these two expressions differ from equation (34) only in the disappearance of the term $g_1(dr/dg)$ from both equations (and the somewhat different interest rate derivative). Thus the effect of a debt swap becomes:

$$\frac{dU}{dg_2} - \frac{dU}{dg_1} = -\frac{\partial U}{\partial e^1}\left(\frac{dr}{dg_2} - \frac{dr}{dg_1}\right)\left(k + g_2 - \frac{s}{1 + r}\right),$$

which depends in sign solely on whether the difference between the interest and growth rates is increased.

The change in the equilibrium interest rates can be derived from the locus of equilibria:

$$r = f'\left(\frac{s(w - (\rho - n)g_1 - (r - n)g_2, r)}{1 + n} - g_2\right)$$

$$\frac{dr}{dg_1} = \frac{-f''\dfrac{\partial s}{\partial w}\left((\rho - n) + g_1\dfrac{d\rho}{dg_1}\right)}{1 + n - f''\dfrac{\partial s}{\partial r} + f''(k + g_2)\dfrac{\partial s}{\partial w}}$$

$$\frac{dr}{dg_2} = \frac{-f''\left(1 + n + (r - n)\dfrac{\partial s}{\partial w}\right)}{1 + n - f''\dfrac{\partial s}{\partial w} + f''(k + g_2)\dfrac{\partial s}{\partial w}}$$

Again assuming that $\rho = r$ and $d\rho/dg = 0$, these derivatives are qualitatively the same as those described in the text.

References

1. W. G. Bowen, R. G. Davis, and D. H. Kopf, "The Public Debt: A Burden on Future Generations?" *Am. Econ. Rev.*, Sept. 1960, *50*, 701–6.
2. T. C. Koopmans, *Three Essays on the State of the Economic Science.* New York 1957.
3. E. J. Mishan, "How to Make a Burden of the Public Debt," *Jour. Pol. Econ.*, Dec. 1963, *71*, 529–42.
4. F. Modigliani, "Long-Run Implications of Alternative Fiscal Policies and the Burden of the National Debt," *Econ. Jour.*, Dec. 1961, *71*, 730–55.
5. R. A. Musgrave, *The Theory of Public Finance.* New York 1959.
6. E. S. Phelps, "The Golden Rule of Accumulation: A Fable for Growthmen," *Am. Econ. Rev.*, Sept. 1961, *51*, 638–43.
7. ———, "Second Essay on the Golden Rule," Cowles Foundation Discussion Paper 173.
8. P. A. Samuelson, "An Exact Consumption-Loan Model of Interest with or without the Social Contrivance of Money," *Jour. Pol. Econ.*, Dec. 1958, *66*, 467–82.
9. ———, "Parable and Realism in Capital Theory: The Surrogate Production Function," *Rev. Econ. Stud.*, June 1962, *29*, 193–206.
10. W. Vickrey, "The Burden of the Public Debt: Comment," *Am. Econ. Rev.*, March 1961, *51*, 132–37.