# Master APE - Economics of Inequalities

Final Exam, February 15, 2012 - 14h30-16h30. No document allowed.

## Exercise 1: Capital Share and Capital/Output Ratio (6 points)

Consider a closed economy with two inputs, raw labor $L$ and capital $K$ and a Cobb-Douglas production function $Y = F(K, L) = K^{\alpha}L^{1-\alpha}$.

1) Show that for any interest rate $r$ and wage rate $w$, the share of income accruing to capital is $\alpha$ (that is, $Y_K = \alpha Y$) (1 point).

2) Show that the elasticity of substitution between labor and capital is equal to 1. (1 point)

3) Does the assumption of a Cobb-Douglas production function accord well with the long-run evolution of the capital/output ratio and the capital share in France and other developed economies? Why? (1 point).

4) Explain how the share of income accruing to capital can be computed using national accounts data (1 point).

5) Express the steady-state capital/output ratio $\beta^* = (K/Y)^*$ as a function of $s$ and $g$ (1 point).

6) In 2010, what is the value of the capital/output ratio $\beta$ and of the net-of-depreciation capital share $\alpha$ in France? What does this imply for the average rate of return to capital in France? (1 point).

## Exercise 2: Pareto coefficients (4 points)

The distributions of high incomes (and high wealth) are usually well approximated by Pareto distributions. That is, above a certain threshold $y_{min}$, the distribution of high incomes $y$ has a cumulative distribution function $\phi(y)$ such that:

$$1 - \phi(y) = (y_{min}/y)^a$$

.

1) We define $b$ the inverted-Pareto coefficient as: $b = \frac{a}{a-1}$. What are the typical values for $b$ for top income distributions in developed economies? (1 point).

2) Explain how $b$ relates to the average income above $y_{min}$ (1 point).

3) How unequal becomes the distribution of $y$ when $b$ tends towards plus infinity? When $b$ tends towards 1? Give the intuition (1 point).

4) Consider the distribution of top incomes in the U.S. How has $b$ evolved over the twentieth century? (1 point).

## Exercise 3: Aggregate Inheritance Flow (10 points)

Consider a standard wealth accumulation model with exogenous growth. National income $Y_t$ is given by a net-of-depreciation production function $F(K_t, H_t)$ where $K_t$ is (non-human) capital, $H_t = L_t e^{gt}$ is human capital (efficient labor), $g$ is the exogenous rate of productivity growth, and $L_t$ is labor supply (raw labor). We consider a closed economy and assume away government assets and liabilities, so private wealth $W_t$ is equal to the capital stock $K_t$. We define $\beta_t = W_t/Y_t$ the wealth/income ratio. Each year, decedents leave $B_t$ in inheritance to their heirs. The goal of this exercise is to study what determines the steady-state inheritance/national income ratio $b_{yt} = B_t/Y_t$ and the steady-state inheritance/private wealth ratio $b_{wt} = B_t/W_t$.

## Accounting Equations

1) We define $\mu_t$ the ratio of the average wealth of the decedents to the average wealth of the living. Assume that $\mu_t$ is equal to 1, and that the mortality rate $m_t$ is equal to 2%. What is the inheritance/private wealth ratio $b_{wt}$ equal to? If the wealth-income ratio $\beta_t$ is equal to 5, what is the inheritance/national income ratio $b_{yt}$ equal to? (1 point).

2) In the general case, express $b_{yt}$ as a function of $\mu_t$, $m_t$, and $\beta_t$ (1 point).

## Dynastic model

Assume the following deterministic, stationary, overlapping-generations demographic structure. Everybody becomes adult at age $a = A$, has exactly one child at age $a = H > A$, and dies at age $D > H$. As a consequence, everybody inherits at age $a = I = D - H > A$. This is a gender-free population. All wealth is transmitted at death (no inter-vivos gifts). Total adult population $N_t$ includes a mass $N_t(a) = 1$ of individuals of age $a$ ($A \leqslant a \leqslant D$) and is permanently equal to $D - A$. The adult mortality rate $m_t$ is also stationary and is given by: $m_t = m^* = 1/(D - A)$.

Example 1: around 1900, we have $A = 20$, $H = 30$, and $D = 60$, so that people inherit at age $I = D - H = 30$. In steady-state, $m^* = 1/(D - A) = 2.5\%$.

Example 2: around 2020, we have $A = 20$, $H = 30$, and $D = 80$, so that people inherit at age $I = D - H = 50$. In steady-state, $m^* = 1/(D - A) = 1.7\%$.

Individuals $i$ born in year $x_i$ maximize an infinite-horizon utility function:

$$U_i = \int_{t \geqslant s} e^{-\theta t} u(c_{ti}) dt$$

with $u(c) = c^{1-\sigma}/(1 - \sigma)$. For simplicity, we assume that individuals start consuming only when they become adult ($s = x_i + A$) and start caring about their children's consumption levels only after they die. We also assume that young adults cannot borrow against their future inheritance, so until age $I$ they can only consume their labor income.

3) In this dynastic model, explain why the steady-state rate of return $r^*$ is $r^* = \theta + \sigma g$ (1 point).

4) In the steady state of the dynastic model, is $r > g$? Why? (1 point).

5) Explain why in the dynastic model the marginal propensity to save out of labor income $s_L$ is equal to 0, and why the marginal propensity to save out of capital income $s_K$ is $g/r^*$ (1 point).

6) Show that at time $t$, individuals aged $a > I$ who have inherited $a - I$ years ago (at time $s = t - a + I$) have the same wealth as individuals who inherit at time $t$ (that is, aged $a = I$ at time $t$) (1 point).

7) Draw the steady-state cross-sectional age-wealth profile, that is the average wealth $w_a(t)$ as a function of $a$ for each $a$ between $A$ and $D$ (1 point). Hint:

- What is the wealth of individuals aged $a < I$?
- What is the wealth of individuals aged $a = I$?
- What is the wealth of individuals aged $a > I$?

8) Compute the average wealth $w_t$ over all age groups $a \in [A, D]$ (1 point).

9) Express $\mu^* = w_t(D)/w_t$ as a function of demographic parameters $A, D, H$ (0.5 point).

10) Show that $b_w^* = 1/H$ and $b_y^* = \beta^*/H$ (0.5 point).

11) Compute $\mu^*$ and $b_y^*$ for two sets of demographic parameters:

- Demography around 1900: $A = 20$, $H = 30$, and $D = 60$
- Demography around 2020: $A = 20$, $H = 30$, and $D = 80$

Interpret the results (1 point).