Master APE - Economics of Inequalities

Final Exam, February 15, 2012 - 14h30-16h30. No document allowed.

Exercise 1: Capital Share and Capital/Output Ratio (6 points)

Consider a closed economy with two inputs, raw labor L and capital K and a Cobb-Douglas production function $Y = F(K, L) = K^{\alpha} L^{1-\alpha}$.

1) Show that for any interest rate r and wage rate w, the share of income accruing to capital is α (that is, $Y_K = \alpha Y$) (1 point).

Answer Take r and w as given. Profit maximization leads to $F_K = r$ and $F_L = w$. $F_K = r$ means $\alpha K^{\alpha-1} L^{1-\alpha} = r$, i.e. $\alpha Y/K = r$, i.e. $Y_K = rK = \alpha Y$.

2) Show that the elasticity of substitution between labor and capital is equal to 1. (1 point)

Answer The elasticity of substitution is: $\epsilon = \frac{dln(L/K)}{dMRTS_{K,L}}$. Remember that $MRTS_{K,L} = \frac{F_K}{F_L}$. Straightforward algebra yields $\epsilon = 1$.

3) Does the assumption of a Cobb-Douglas production function accord well with the long-run evolution of the capital/output ratio and the capital share in France and other developed economies? Why? (1 point).

Answer Historically, the capital share was lower when capital/output was lower. This suggests that the elasticity of substitution is above 1, hence that the

Cobb-Douglas production function is not fully appropriate.

4) Explain how the share of income accruing to capital can be computed using national accounts data (1 point).

Answer Total net capital income = Gross profits + share of capital income in self-employment income - capital depreciation.

Three methods to compute the capital share of self-employment income: (1) apply the same capital share as in the corporate sector; (2) apply the same average labor income as in the corporate sector (ideally controlling for skills: wage equation in employment survey); (3) apply the same rate of return to capital stock as in the corporate sector.

5) Express the steady-state capital/output ratio $\beta^* = (K/Y)^*$ as a function of s and g (1 point).

Answer

$$\dot{\beta} = \dot{K}/K - \dot{Y}/Y = sY/K - g = 0$$

$$\beta^* = s/g$$

.

6) In 2010, what is the value of the capital/output ratio β^* and of the net-of-depreciation capital share α in France? What does this imply for the average rate of return on capital in France? (1 point).

Answer $\beta^* = 600\%$ and $\alpha = 30\%$ so $r = \alpha/\beta = 5\%$.

Exercise 2: Pareto coefficients (4 points)

The distributions of high incomes are usually well approximated by Pareto distributions. That is, above a certain threshold y_{min}, the distribution of high incomes

y has a cumulative density function $\phi(y)$ such that:

$$1 - \phi(y) = (y_{min}/y)^a$$

.

1) We define b the inverted-Pareto coefficient as:

$$b = \frac{a}{a-1}$$

What are the typical values for b for top income distributions in developed economies? (1 point).

Answer $b \approx 2$.

2) Explain how b relates to the average income above y_{min} (no proof required) (1 point).

Answer $b = \bar{y}/y_{min}$.

3) How unequal becomes the distribution of y when b tends towards plus infinity? When b tends towards 1? (1 point).

Answer b tends towards plus infinity: infinite inequality. b tends towards 1: no inequality.

4) Consider the distribution of top incomes in the U.S. How has b evolved over the twentieth century? (1 point).

Answer The top income share decreased in the 30s-40s; then increased in the 80s-90s-2000s. Similar U-shaped pattern for b.

Exercise 3: Aggregate Inheritance Flow (10 points)

Consider a standard wealth accumulation model with exogenous growth. National income Y_t is given by a net-of-depreciation production function $F(K_t, H_t)$ where K_t is (non-human) capital, $H_t = L_t e^{gt}$ is human capital (efficient labor), g is the exogenous rate of productivity growth, and L_t is labor supply (raw labor). We consider a closed economy and assume away government assets and liabilities, so private wealth W_t is equal to the capital stock K_t. We define $\beta_t = W_t/Y_t$ the wealth/income ratio. Each year, decedents leave B_t in inheritance to their heirs. The goal of this exercise is to study what determines the steady-state inheritance/national income ratio $b_{yt} = B_t/Y_t$ and the steady-state inheritance/private wealth ratio $b_{wt} = B_t/W_t$.

Accounting Equations

1) We define μ_t the ratio of the average wealth of the decedents to the average wealth of the living. Assume that μ_t is equal to 1, and that the mortality rate m_t is equal to 2%. What is the inheritance/private wealth ratio b_{wt} equal to? (0.5 point)? If the wealth-income ratio β_t is equal to 5, what is the inheritance/national income ratio b_{yt} equal to? (0.5 point).

Answer $b_{wt} = 2\%$ and $b_{yt} = 10\%$.

2) In the general case, express b_{yt} as a function of μ_t, m_t, and β_t (1 point).

Answer $b_{yt} = m_t \times \mu_t \times \beta_t$.

Dynastic model

Assume the following deterministic, stationary, overlapping-generations demographic structure. Everybody becomes adult at age $a = A$, has exactly one child at age $a = H > A$, and dies at age $D > H$. As a consequence, everybody inherits at age $a = I = D - H > A$. This is a gender-free population. All wealth is transmitted at death (no inter-vivos gifts). Total adult population N_t includes a mass $N_t(a) = 1$ of individuals of age a ($A \leqslant a \leqslant D$) and is permanently equal to $D - A$. The adult mortality rate m_t is also stationary and is given by: $m_t = m^* = 1/(D - A)$.

Example: around 2020, we have $A = 20$, $H = 30$, and $D = 80$, so that people inherit at age $I = D - H = 50$. In steady-state, $m^* = 1/(D - A) = 1.7\%$.

Individuals i born in year x_i maximize an infinite-horizon utility function:

$$U_i = \int_{t \geqslant s} e^{-\theta t} u(c_{ti}) dt$$

with $u(c) = c^{1-\sigma}/(1 - \sigma)$. For simplicity, we assume that individuals start consuming only when they become adult ($s = x_i + A$) and start caring about their children's consumption levels only after they die. We also assume that young adults cannot borrow against their future inheritance, so until age I they can only consume their labor income.

3) In this dynastic model, explain why the steady-state rate of return r^* is:

$$r^* = \theta + \sigma g$$

(1 point).

Answer: This follows directly from the first-order condition describing the optimal consumption path: $dc_t/dt = (r - \theta)c_t/\sigma$ i.e. utility-maximizing agents want their consumption path to grow at rate $g_c = (r - \theta)/\sigma$. This is a steady-state if and only if $g_c = g$, i.e. $r = r^* = \theta + \sigma g$. If $r > r^*$ they accumulate indefinitely, and if $r < r^*$ they borrow indefinitely.

4) In the steady state of the dynastic model, is $r > g$? Why? (1 point).

Answer: σ is the inverse of the inter temporal elasticity of substitution (IES). Realistic values for the IES are usually considered to be relatively small (typically between 0.2 and 0.5), and in any case smaller than one, so σ is a parameter that is typically bigger than one, so $r > g$.

5) Explain why in the dynastic model the marginal propensity to save out of labor income s_L is equal to 0, and why the marginal propensity to save out of capital income s_K is g/r^* (1 point).

Answer: In the steady state of the dynastic model, the consumption path of every dynasty (poor or rich) grows at rate g. Since labor income naturally grows at rate g, zero-wealth dynasties do not need to save out of labor income. However wealth does not naturally grow at rate g. So if wealthy dynasties do not save, and instead consume the full return to their inherited wealth, then their future consumption will not grow. In order to make sure that their wealth and future capital income grows at rate g, they need to save a fraction $s_K = g/r^*$. Now, because $r^* > g$, $s_K = g/r^* < 100\%$: wealthy dynasties consume a positive fraction 1-g/r* of the return to their inherited wealth and save the rest.

6) Show that at time t, individuals aged $a > I$ who have inherited $a - I$ years ago (at time $s = t - a + I$) have the same wealth as individuals who inherit at time t (that is, aged $a = I$ at time t) (1 point).

Answer: Individuals with age $a > I$ at time t inherited $a - I$ years ago, at time $s = t - a + I$. They received average bequests $b_s = w_s(I)$ that are smaller than the average bequests b_t inherited at time t by the I-year-old. Because everything grows at rate g in steady state, we have $b_s = e^{-g(a-I)}b_t$. Although they received smaller bequests, they saved a fraction $s_K = g/r^*$ of the corresponding return, so at time t their inherited wealth is now equal to $w_t(a) = e^{s_K r(a-I)}e^{-g(a-I)}b_t = b_t$.

7) Draw the steady-state cross-sectional age-wealth profile, that is the average wealth $w_a(t)$ as a function of a for each a between A and D (1 point). Hint:

- What is the wealth of individuals aged $a < I$?
- What is the wealth of individuals aged $a = I$?
- What is the wealth of individuals aged $a > I$?

Answer: Since $s_L = 0$, young adults have zero wealth until the time they inherit:

$$\forall a \in [A, I[, w_t(a) = 0$$

Then at age $a = I$, everybody inherits. On average agents inherit $b_t = w_t(I)$. So:

$$w_t(I) = b_t = \bar{w}_t$$

Then all age groups with age between I and D have the same wealth $\bar{w}_t$ in the cross-section (see above):

$$\forall a \in [I, D] w_t(a) = b_t = \bar{w}_t$$

8) Compute the average wealth w_t over all age groups $a \in [A, D]$ (1 point).

Answer: $w_t = (D - I)\bar{w}_t/(D - A)$ (remember that total adult population is permanently equal to $D - A$) $= H\bar{w}_t/(D - A)$ (remember that by definition $H = D - I$).

9) Express $\mu^* = w_t(D)/w_t$ as a function of demographic parameters A, D, H (0.5 point).

Answer: $\mu^* = \frac{D-A}{H}$.

10) Show that $b_w^* = 1/H$ and $b_y^* = \beta^*/H$ (0.5 point).

Answer: $b_{wt}^* = \mu^* m^* = 1/H$ and $b_{yt}^* = \beta^* b_{wt} = \beta^*/H$.

11) Compute μ^* and b_y^* for two sets of demographic parameters:

- Demography around 1900: $A = 20$, $H = 30$, and $D = 60$
- Demography around 2020: $A = 20$, $H = 30$, and $D = 80$

Interpret the results (1 point).

Answer:

- Demography around 1900: $\mu^* = 133\%$ and $b_y^* = 20\%$
- Demography around 2020: $\mu^* = 200\%$ and $b_y^* = 20\%$

In aging societies with higher life expectancy, people die less often, but they die with higher relative wealth, so that the aggregate inheritance flow is unchanged. In effect, the entire wealth profile is imply shifted toward older age groups: one has to wait longer before inheritance, but one inherits larger amounts, so that from a lifetime perspective inheritance is just as important as before.

This exercise was based on the model presented in Section V of Piketty T. (2011), “On the long-run evolution of inheritance: France 1820-2050”, *Quarterly Journal of Economics*, vol. 76, n. 3.